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On the Distribution of Divisors of Integers in Residue Classes (mod k)

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We consider the distribution of the divisors of n among the reduced residue classes (mod k), and establish mean-value theorems for the variance of this distribution.

A result to the effect that for almost all $n < x$ this distribution is very even, provided k is not too large compared with x, is derived; this is similar to but more precise than a theorem of Erdös, and is thus a new proof of his result.

INTRODUCTION

Let $\tau(n; k)$ denote the number of divisors of n which are prime to k, and $\tau(n; k, h)$ the number of these congruent to $h(\text{mod } k)$, for each h, $1 \leqslant h < k, (h, k) = 1$.

The purpose of this paper is to evaluate the sum

$$\sum_{n\leqslant x} y^{\omega(n)} \sum_{\substack{h=1\\(h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2,$$

where y is any complex number and $\omega(n)$ denotes the number of prime factors of n counted according to multiplicity. In Theorems 1, 2, and 4 we do this in the cases $y = 1$, $y = \frac{1}{2}$ and $|y| \leqslant 1$, $\mathbf{R}y < \frac{1}{2}$, respectively, in each case uniformly for as wide a range of values of k as we can.

Erdös [1] proved that for all $\eta > 0$, for almost all $n < x$, and all h prime to k,

$$(1-\eta)\frac{\tau(n; k)}{\varphi(k)} < \tau(n; k, h) < (1+\eta)\frac{\tau(n; k)}{\varphi(k)}$$

provided

$$k < 2^{\frac{1}{2}(1-\epsilon)\log\log x}.$$

His method depends on a theorem of Erdös and Renyi [8] concerning the representation of elements of an Abelian group by products of distinct elements selected from a small prescribed set. He remarks that any improvement in his result with Renyi would lead to a corresponding increase in the range of values of k given above. In our Theorem 5 a similar result is derived, but from Theorem 4 in place of the Erdös–Renyi result. We give an upper bound for the number of exceptional n, in terms of x, η and k, and the result is valid for

$$k < 2^{\frac{1}{2}(1-\{\epsilon_0/\log\log\log\log x\})\log\log x},$$

ϵ_0 being an absolute constant which could be computed. This does not lead to any conclusion about the Erdös–Renyi result, but suggests that it might be best possible, perhaps for cyclic groups at least.

We hope to derive a formula for the sum

$$\sum_{\substack{n\leqslant x\\ \omega(n)=t}} \sum_{\substack{h=1\\ (h,k)=1}}^{k} \left(\tau(n;k,h) - \frac{\tau(n;k)}{\varphi(k)}\right)^2$$

by means of an application of Cauchy's theorem to our Theorem 4, in a later paper. A preliminary study shows that this could throw some light on the conjecture that the Erdös–Renyi theorem is best possible.

Notation. $C_1, C_2, \ldots$, are absolute constants, independent of all parameters except possibly ϵ, when they are written in the form $C_i(\epsilon)$. They are defined to be large enough, or small enough, to ensure the validity of formulae in which they occur. As usual, χ denotes a character mod k, and χ_0 denotes the principal character mod k.

Theorems and Proofs

Theorem 1. For fixed k, as $x \to \infty$,

$$\sum_{n<x} \sum_{\substack{h=1\\ (h,k)=1}}^{k} \left(\tau(n;k,h) - \frac{\tau(n;k)}{\varphi(k)}\right)^2 \sim \frac{6x\log x}{\pi^2\varphi(k)} \prod_{p|k} \left(\frac{p}{p+1}\right) \sum_{\chi\neq\chi_0} |L(1,\chi)|^2.$$

More precisely, if $k < \sqrt{x}$, then the sum on the left is equal to

$$\frac{6x}{\pi^2\varphi(k)} \prod_{p|k} \left(\frac{p}{p+1}\right) \Big[\Big(\log x + 2\gamma - 1 - \frac{\zeta'(2)}{\zeta(2)} + \sum_{p|k} \frac{\log p}{p+1} \Big) \sum_{\chi\neq\chi_0} |L(1,\chi)|^2$$

$$+ \sum_{\chi\neq\chi_0} L'(1,\chi)\, L(1,\bar\chi) + L(1,\chi)\, L'(1,\bar\chi) \Big] + O(x^{4/5+\epsilon}k^{2/5}).$$

The sums over characters χ on the right are evaluated in Lemma 2.

THEOREM 2. For $\log k < (\log x)^{1/4} (\log\log x)^{-3/4}$,

$$\sum_{n<x} \frac{1}{2^{\omega(n)}} \sum_{h=1}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2 = \frac{Hx}{\varphi(k)} \prod_{p|k} \left(\frac{2p}{2p+1}\right) \sum_{\chi \neq \chi_0} | L(1, \tfrac{1}{2}, \chi)|^2 + O\left(x \exp\left(\frac{-C_1(\log x)^{1/4}}{(\log\log x)^{3/4}}\right)\right)$$

where

$$L(s, y, \chi) = \prod_p \left(1 - \frac{y\chi(p)}{p^s}\right)^{-1}$$

and

$$H = H\left(\frac{1}{2}\right) = \lim_{s\to 1} (s-1) \prod_p \left(\frac{2p^s+1}{2p^s-1}\right).$$

THEOREM 3. Let $\sigma_a(n; k)$ be the sum of the ath powers of those divisors of n which are prime to k, and let $\sigma_a(n; k, h)$ be defined similarly. As usual, we write τ and σ for σ_0 and σ_1. Then

$$\sum_{n<x} \frac{1}{n^2} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\sigma(n; k, h) - \frac{\sigma(n; k)}{\varphi(k)}\right)^2 = \frac{90x}{\pi^4 \varphi(k)} \prod_{p|k} \left(\frac{p^4 - p^3}{p^4 - 1}\right) \sum_{\chi \neq \chi_0} | L(2, \chi)|^2 + O(\tau(k)\, x^{1/3+\epsilon}).$$

Reference to Lemma 2 shows that the only dependence on k on the right is a factor of roughly $\varphi(k)/k$. Thus $\sigma(n; k)/\varphi(k)$ is not generally a valid approximation to $\sigma(n; k, h)$ for all h; there are some n and h for which

$$\sigma(n; k, h) > \frac{\sigma(n; k)}{\sqrt{\varphi(k)}}.$$

A similar result holds for all positive powers of the divisors.

THEOREM 4. For all $\epsilon > 0$, and $k^\epsilon < (\log x \log\log x)^{3/4}$, and all complex y for which $| y | \leqslant 1$ and $2\mathbf{R}y < 1$, we have

$$\sum_{n<x} y^{\omega(n)} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2 = \frac{H(y)x}{\Gamma(2y)(\log x)^{1-2y}} \prod_{p|k} \left(\frac{p}{p+y}\right) \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} L(1, y, \chi)\, L(1, y, \bar{\chi}) + O\left(\frac{C_2(\epsilon)\, k^{2\epsilon} x}{(\log x)^{2-2\mathbf{R}y}}\right),$$

where

$$H(y) = \lim_{s\to 1} (s-1)^{2y} \prod_p \left(\frac{p^s + y}{p^s - y}\right).$$

THEOREM 5. There exists an absolute constant $\epsilon_0 > 0$ such that for

$$k \leqslant 2^{\frac{1}{2}(1-\{\epsilon_0/\log\log\log\log x\})\log\log x}$$

we have, for any $\eta > 0$, for almost all $n \leqslant x$, all h prime to k,

$$(1-\eta)\frac{\tau(n;k)}{\varphi(k)} < \tau(n;k,h) < (1+\eta)\frac{\tau(n;k)}{\varphi(k)}.$$

The number of exceptional n does not exceed

$$C_{34} \times \left[\frac{1}{(\log x)^{\epsilon^2/4}} + \frac{1}{\eta\sqrt{\epsilon}\,(\log x)^{\epsilon/8}}\right],$$

where ϵ is given by

$$k^2 = 2^{(1-\epsilon)\log\log x}.$$

The proofs of these theorems are based on a generalization of the well-known identity of Ramanujan [2],

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\,\sigma_b(n)}{n^s} = \frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

This generalization is contained in the following lemma.

LEMMA 1. Let a, b, and y be any complex numbers, $|y| \leqslant 1$. Let

$$\sigma_a(n,\chi) = \sum_{d|n} \chi(d)\,d^a,$$

and

$$L(s,y,\chi) = \prod_p \left(1 - \frac{y\chi(p)}{p^s}\right)^{-1}, \qquad \zeta(s,y) = \prod_p \left(1 - \frac{y}{p^s}\right)^{-1}.$$

Let $\omega(n)$ be the number of prime factors of n, counted according to multiplicity. Then

$$\sum_{n=1}^{\infty} y^{\omega(n)} \frac{\sigma_a(n,\chi_1)\,\sigma_b(n,\chi_2)}{n^s}$$

$$= \frac{\zeta(s,y)\,L(s-a,y,\chi_1)\,L(s-b,y,\chi_2)\,L(s-a-b,y,\chi_1\chi_2)}{L(2s-a-b,y^2,\chi_1\chi_2)}$$

provided

$$\min \mathbf{R}\{s, s-a, s-b, s-a-b\} > 1.$$

We do not give the proof which is similar to that of Ramanujan's formula.

Ingham [6] first used an identity of this type to give his elegant proof that

$$L(1+it, \chi) \neq 0.$$

The advantage of Ingham's method is that it is quite general and covers the case $t = 0$, χ real. We use this identity as follows. We have

$$\sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\sigma_a(n; k, h) - \frac{\sigma_a(n; k)}{\varphi(k)}\right)^2 = \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \sigma_a^2(n; k, h) - \frac{1}{\varphi(k)} \sigma^2(n; k)$$

$$= \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} \sigma_a(n, \chi)\, \sigma_a(n, \bar{\chi}).$$

Thus, by Lemma 1,

$$\sum_{n=1}^{\infty} \frac{y^{\omega(n)}}{n^s} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\sigma_a(n; k, h) - \frac{\sigma_a(n; k)}{\varphi(k)}\right)^2$$

$$= \frac{1}{\varphi(k)} \frac{\zeta(s, y)\, L(s-2a, y, \chi_0)}{L(2s-2a, y^2, \chi_0)} \sum_{\chi \neq \chi_0} L(s-a, y, \chi)\, L(s-a, y, \bar{\chi}).$$

When $a = 0$, $y = 1$, there is a double pole at $s = 1$, instead of the quadruple pole which arises in the calculation of the average order of $\tau^2(n; k)$. For positive a, there is no such saving, the simple pole at $s = 1 + 2a$ being unaltered by the introduction of nonprincipal characters. Thus we expect $a = 0$ to be the most interesting case.

Proof. of Theorem 1. Clearly the result for fixed k follows from the more exact uniform result. We set

$$W_k(n) = \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2$$

and we observe that, for any $\epsilon_1 > 0$, $W_k(n) \leqslant \tau^2(n) = O(n^{\epsilon_1})$. Also

$$\sum_{n=1}^{\infty} \frac{W_k(n)}{n^\sigma} \leqslant \sum_{n=1}^{\infty} \frac{\tau^2(n)}{n^\sigma} = O\left(\frac{1}{(\sigma-1)^4}\right) \quad \text{as} \quad \sigma \to 1+.$$

Lemma 3.12 of Titchmarsh [3] gives

$$\sum_{n<x} W_k(n) = \frac{1}{2i\pi}\int_{c-iT}^{c+iT} \frac{x^s}{s} F(s)\, ds + O\left(\frac{x^c}{T(c-1)^4}\right) + O\left(\frac{x^{1+2\epsilon_1}}{T}\right)$$

where x is half an odd integer, and

$$F(s) = \frac{1}{\varphi(k)} \prod_{p|k}\left(\frac{p^s}{p^s+1}\right) \frac{\zeta^2(s)}{\zeta(2s)} \sum_{\chi\neq\chi_0} L(s,\chi)\, L(s,\bar{\chi}).$$

We move the contour of integration onto the other three sides of the rectangle $c+iT$, $a+iT$, with $\frac{1}{2} < a \leqslant \frac{3}{4}$. We encounter a double pole at $s=1$ with residue the main term given in the theorem. We take $c = 1 + (1/\log x)$. On the line $a+it$ we have (see Prachar [7])

$$\begin{aligned}
\zeta(s) &= O(|\,t\,|^{1/2(1-a)+\epsilon_2}) && |\,t\,| > a,\\
L(s,\chi) &= O((k\,|\,t\,|)^{1-a}) && |\,t\,| > a,\\
1/\zeta(2s) &= O\left(\frac{1}{2a-1}\right) && \text{for all } t,\\
\zeta(s) &= O(1) && |\,t\,| \leqslant a,\\
L(s,\chi) &= O(k^{1/2}) && |\,t\,| \leqslant a,
\end{aligned}$$

and

$$\prod_{p|k}\left(\frac{p^s}{1+p^s}\right) = O(\tau(k)),$$

where here $\chi \neq \chi_0$. Thus

$$\frac{1}{2i\pi}\int_{a-iT}^{a+iT} \frac{x^s}{s} F(s)\, ds = O\left(\frac{\tau(k)\, x^a T^{3(1-a)+2\epsilon_2} k^{2(1-a)}}{2a-1}\right).$$

The integrals over the lines parallel to the σ axis do not exceed

$$O\left(\int_a^1 \frac{x^\sigma}{T}\cdot\frac{T^{3(1-\sigma)+2\epsilon_2}k^{2(1-\sigma)}\tau(k)}{2a-1}\, d\sigma + \int_1^c \frac{x^c}{T}\,\tau(k)\log^4 kT\, d\sigma\right)$$

$$= O\left(\frac{k^2T^{2(1+\epsilon_2)}\tau(k)}{(2a-1)}\left(\frac{x}{k^2T^3} + \frac{x^a}{k^{2a}T^{3a}}\right) + \frac{x\tau(k)\log^4 kT}{T\log x}\right).$$

Thus the integral around the three sides of the rectangle does not exceed

$$O\left(\frac{\tau(k)\, x^a k^{2(1-a)} T^{3(1-a)+2\epsilon_2}}{2a-1} + \frac{x\tau(k)\, T^{2\epsilon_2}}{T(2a-1)}\right).$$

These two terms are of the same order if

$$T = \left(\frac{x}{k^2}\right)^{\frac{1-a}{1+3(1-a)}}.$$

We select

$$a = \tfrac{1}{2} + (1/\log x)$$

and find that the integral does not exceed

$$C_3 x^{4/5} k^{2/5} \tau(k) \log x \cdot \left(\frac{x}{k^2}\right)^{\epsilon_2}.$$

Hence for all $\epsilon > 0$, $k^2 < x$,

$$\sum_{n \leqslant x} W_k(n) = \mathbf{Res}(1) + O(x^{4/5+\epsilon} k^{2/5});$$

the first term on the right being the residue at $s = 1$. This completes the proof.

Proof of Theorem 2. By Lemma 1, we have, for $|y| \leqslant 1$,

$$\sum_{n=1}^{\infty} \frac{y^{\omega(n)}}{n^s} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2$$

$$= F(s) = \frac{1}{\varphi(k)} \prod_{p|k} \left(\frac{p^s}{p^s + y}\right) \frac{\zeta^2(s, y)}{\zeta(2s, y^2)} \sum_{\chi \neq \chi_0} L(s, y, \chi)\, L(s, y, \bar{\chi}).$$

It is plain that

$$L(s, y, \chi) = \{L(s, \chi)\}^y H(s, y, \chi)$$

where H is regular and bounded independently of k in the half plane $\mathbf{R}s \geqslant \frac{1}{2} + \delta$. Thus $F(s)$ is regular in the zero-free region of L functions to modulus k except for an algebraical singularity at $s = 1$. There are two important differences between the special case $y = \frac{1}{2}$ and the general case $|y| \leqslant 1$. First, when $y = \frac{1}{2}$, $F(s)$ is regular at the (possible) Siegel zero of $L(s, \chi)$. For χ is real and so

$$L(s, \chi)^y L(s, \chi)^y = L(s, \chi)^{2y} = L(s, \chi).$$

Also, when $y = 1/2$, F has a simple pole at $s = 1$.

Set $c = 1 + 1/\log x$, x half an odd integer. By Lemma 3.12 of Titchmarsh [3] we have

$$\sum_{n<x} \frac{1}{2^{\omega(n)}} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2$$
$$= \frac{1}{2i\pi} \int_{c-iT}^{c+iT} \frac{x^s}{s} F(s)\, ds + O\left(\frac{x}{T} 2^{\frac{(1+\epsilon)\log x}{\log\log x}}\right).$$

The error term on the right arises because our best estimate for the coefficients of the Dirichlet series on the left is

$$\frac{1}{2^{\omega(n)}} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2 \leqslant \frac{\tau^2(n)}{2^{\omega(n)}} \leqslant 2^{(1+\epsilon)\frac{\log n}{\log\log n}}.$$

Thus we have to select $\log T$ to be at least of order $\log x/\log\log x$, and

$$x^{1-(A/\log kT)} > x/(\log x)^A,$$

so that an estimation of the above integral obtained by moving the contour a distance of only $A/\log k \mid t \mid$ to the left would be ineffective. We move the contour to

$$\sigma = 1 - \frac{C_4}{M(k, t)},$$

where

$$M(k, t) = \max(\log k, \log^{3/4}(\mid t \mid + 3)\{\log\log(\mid t \mid + 3)\}^{3/4}),$$

the contour being completed by horizontal lines at $\mid t \mid = T$. On this curve we have, (see Prachar [7]),

$$\mid L(s, \chi) \mid \ll \exp\left\{\log k \frac{\{\log\log(\mid t \mid + 3)\}^{9/4}}{\{\log(\mid t \mid + 3)\}^{3/4}} + C_4(\log\log(\mid t \mid + 3))^3\right\}.$$

The residue of $F(s)$ at $s = 1$ is

$$\frac{Hx}{\varphi(k)} \prod_{p \mid k} \left(\frac{p}{2p+1}\right) \sum_{\chi \neq \chi_0} \mid L(1, \tfrac{1}{2}, \chi) \mid^2,$$

and the integral over the new contour does not exceed

$$O\left(k^\epsilon \int_0^T \frac{x^{1-c_4/M(k,t)}}{1+t} \exp\left\{\log k \frac{\{\log\log(\mid t \mid + 3)\}^{9/4}}{\{\log(\mid t \mid + 3)\}^{3/4}} + C_6(\log\log T)^3\right\} dt\right.$$
$$\left. + \frac{k^\epsilon x^c}{T} \exp\left\{\log k \frac{(\log\log T)^{9/4}}{(\log T)^{3/4}} + C_6(\log\log T)^3\right\}\right).$$

If

$$\log k < (\log T \cdot \log\log T)^{3/4}$$

this does not exceed

$$O\left(k^{\epsilon}\left\{\frac{x^c}{T} + x\exp\left(-\frac{C_4 \log x}{(\log T)^{3/4}(\log\log T)^{3/4}}\right)\right\}\right.$$
$$\left.\times \exp\left\{\log k\,\frac{(\log\log T)^{9/4}}{(\log T)^{3/4}} + C_6(\log\log T)^3\right\}\right).$$

We set $T = x$, $c = 1 + (1/\log x)$, and deduce that for

$$\log k < (\log x)^{1/4}(\log\log x)^{-3/4}$$

$$\sum_{n<x}\frac{1}{2^{\omega(n)}}\sum_{\substack{h=1\\(h,k)=1}}^{k}\left(\tau(n;k,h) - \frac{\tau(n;k)}{\varphi(k)}\right)^2$$
$$= \frac{Hx}{\varphi(k)}\prod_{p\mid k}\left(\frac{2p}{2p+1}\right)\sum_{\chi\neq\chi_0}|\,L(1,\tfrac{1}{2},\chi)|^2$$
$$+ O\left(x\exp\left(-\frac{C_1(\log x)^{1/4}}{(\log\log x)^{3/4}}\right)\right).$$

This completes the proof.

Lemma 2.

$$\frac{1}{\varphi(k)}\sum_{\chi\neq\chi_0}|\,L(1,\chi)|^2 = \frac{\pi^2}{6}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right) + O\left(\frac{\log^2 k}{k}\right), \tag{1}$$

$$\frac{1}{\varphi(k)}\sum_{\chi\neq\chi_0}|\,L(2,\chi)|^2 = \frac{\pi^4}{90}\prod_{p\mid k}\left(1-\frac{1}{p^4}\right) + O\left(\frac{1}{\varphi(k)}\right), \tag{2}$$

$$\frac{1}{\varphi(k)}\sum_{\chi\neq\chi_0} L(1,\chi)\,L'(1,\bar{\chi}) + L'(1,\chi)\,L(1,\bar{\chi})$$
$$= \frac{\pi^2}{3}\prod_{p\mid k}\left(1-\frac{1}{p^2}\right)\left(\frac{\zeta'(2)}{\zeta(2)} + \sum_{p\mid k}\frac{\log p}{p^2-1}\right) + O\left(\frac{\log^3 k}{\varphi(k)}\right). \tag{3}$$

The first result, due to H.–E. Richert, appears in a book by Halberstam & Richert which is in the course of publication, and I am grateful to Professor Halberstam for pointing it out to me. We therefore have to prove (2) and (3), and our method gives (1) with a slightly weaker error term.

Proof. Set

$$F_k(s) = \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} L(s, \chi)\, L(s, \bar{\chi}).$$

For $\mathbf{R}s > 1$, we have

$$\begin{aligned} F_k(s) &= \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \Big[\sum_{\substack{n=1 \\ n \equiv h}}^{\infty} \frac{1}{n^s} - \frac{1}{\varphi(k)} \sum_{(n,k)=1} \frac{1}{n^s} \Big]^2 \\ &= \sum_h \frac{1}{\varphi^2(k)} \Big[\sum_{\substack{l=1 \\ (l,k)=1}}^{k} \sum_{r=0}^{\infty} \frac{1}{(rk+h)^s} - \frac{1}{(rk+l)^s} \Big]^2. \end{aligned}$$

Both sides are regular for $\mathbf{R}s > 0$ and so, by analytic continuation, equal. Hence we have

$$\begin{aligned} F_k'(1) &= \frac{-2}{\varphi^2(k)} \sum_h \Big[\sum_l \sum_r \Big(\frac{1}{rk+h} - \frac{1}{rk+l} \Big) \Big] \\ &\quad \times \Big[\sum_l \sum_r \Big(\frac{\log(rk+h)}{rk+h} - \frac{\log(rk+l)}{rk+l} \Big) \Big] \\ &= \frac{-2}{\varphi^2(k)} \sum_h \Big[\frac{\varphi(k)}{h} + O(\log k) \Big] \Big[\frac{\varphi(k) \log h}{h} + O(\log^2 k) \Big] \\ &= \frac{-2}{\varphi^2(k)} \sum_h \Big[\frac{\varphi^2(k) \log h}{h^2} + O\Big(\frac{\varphi(k) \log^2 k}{h} + \log^3 k \Big) \Big] \\ &= 2 \frac{d}{ds} \Big\{ \zeta(s) \prod_{p \mid k} \Big(1 - \frac{1}{p^s} \Big) \Big\} \Big|_{s=2} + O\Big(\frac{\log^3 k}{\varphi(k)} \Big) \\ &= \frac{\pi^2}{3} \prod_{p \mid k} \Big(1 - \frac{1}{p^2} \Big) \Big(\frac{\zeta'(2)}{\zeta(2)} + \sum_{p \mid k} \frac{\log p}{p^2 - 1} \Big) + O\Big(\frac{\log^3 k}{\varphi(k)} \Big). \end{aligned}$$

Similarly,

$$\begin{aligned} F_k(2) &= \frac{1}{\varphi^2(k)} \sum_h \Big[\frac{\varphi(k)}{h^2} + O(1) \Big]^2 \\ &= \zeta(4) \prod_{p \mid k} \Big(1 - \frac{1}{p^4} \Big) + O\Big(\frac{1}{\varphi(k)} \Big). \end{aligned}$$

This completes the proof.

Proof of Theorem 3. We set

$$H_k(n) = \frac{1}{n^2} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \Big(\sigma(n; k, h) - \frac{\sigma(n; k)}{\varphi(k)} \Big)^2 \leqslant \frac{\sigma^2(n)}{n^2}.$$

We have, by Lemma 1 and Lemma 3.12 of Titchmarsh [3],

$$\sum_{n<x} H_k(n) = \frac{1}{2i\pi}\int_{c-iT}^{c+iT} \frac{x^s}{s} f(s)\,ds + O\left(\frac{x^c}{T(c-1)}\right) + O\left(\frac{x\log^2 x}{T}\right)$$

where

$$f(s) = \frac{1}{\varphi(k)} \frac{\zeta(s+2)\,L(s,\chi_0)}{L(2s+2,\chi_0)} \sum_{\chi\neq\chi_0} L(s+1,\chi)\,L(s+1,\bar{\chi}),$$

provided x is half an odd integer. Setting $c = 1 + 1/\log x$, and replacing the contour of integration by the other three sides of the rectangle $a + iT$, $c + iT$, we have

$$\begin{aligned}\sum_{n<x} H_k(n) = {}& \frac{x}{\varphi(k)} \cdot \frac{1}{\zeta(4)} \prod_{p\mid k}\left(\frac{p^4-p^3}{p^4-1}\right) \sum_{\chi\neq\chi_0} \mid L(2,\chi)|^2 + O\left(\frac{x\log^2 x}{T}\right)\\ &+ O\left(\int_0^a \frac{x^a}{a}\,\tau(k)\,\zeta^2(1+a)\,dt\right.\\ &+ \int_a^T \frac{x^a}{t}\,\tau(k)\,t^{1/2(1-a)+\epsilon_1}\zeta^2(1+a)\,dt\\ &\left.+ \int_a^c \frac{x^\sigma}{T}\,\tau(k)\,T^{1/2(1-\sigma)+\epsilon_1}\zeta^2(1+a)\,d\sigma\right).\end{aligned}$$

Provided $x > \sqrt{T}$, the integrand in the last term is increasing. Hence the total error term is

$$O\left(\frac{x\log^2 x}{T} + \frac{x^a}{a^2}\,\tau(k)\,T^{1/2(1-a)+\epsilon_1} + \frac{x}{a^2 T}\,\tau(k)\,T^{\epsilon_1}\right).$$

Setting $T = x^{2/3}$, $a = \epsilon_1 = \epsilon/2$, the result follows.

LEMMA 3. Let

$$M(k,t) = \max\{\log k, \{\log(|\,t\,| + 3)\log\log(|\,t\,| + 3)\}^{3/4}\}.$$

Then it is known (see Prachar [7]), that

$$L(s,\chi) \neq 0$$

in the region

$$\sigma \geqslant 1 - \frac{2C_7}{M(k,t)}$$

provided $\chi \neq \chi_1$, the possible real character with a real (Siegel) zero. If $\chi = \chi_1$, the same result holds for $t \neq 0$. For $t = 0$, $L(s,\chi_1) \neq 0$ for

$$\sigma \geqslant 1 - \frac{2C_8(\epsilon)}{k^\epsilon}$$

for fixed $\epsilon > 0$. In the region

$$\sigma \geqslant 1 - \frac{C_7}{M(k, |t| + 1)}, \qquad |t| \geqslant 2,$$

we have, for nonprincipal characters,

$$|\operatorname{Log} L(s, \chi)| \leqslant C_9 \log k \frac{\{\log\log(|t| + 3)\}^{9/4}}{\{\log(|t| + 3)\}^{3/4}} + C_{10}\{\log\log(|t| + 3)\}^3 + C_{11} \log\log 2k.$$

For $|t| < 2$, $\chi \neq \chi_0$ or χ_1,

$$|\operatorname{Log} L(s, \chi)| \leqslant C_{12} \log\log 2k,$$

and for $\chi = \chi_1$,

$$|\operatorname{Log} L(s, \chi)| \leqslant \tfrac{1}{2} \log k + C_{13}(\epsilon)$$

both in the region

$$\sigma \geqslant 1 - \frac{C_8(\epsilon)}{4k^\epsilon},$$

provided $\epsilon \leqslant 3/11$.

Proof. Let r and R be positive real numbers, and let $s_0 = s + r$ have real part $\sigma_0 > 1$. Let $R > r$ and $A(R)$ denote the maximum of $\operatorname{Log} |L(s, \chi)|$ on the circle centre s_0, radius R. We set

$$R = \sigma_0 - 1 + \frac{2C_7}{M(k, |t| + 1)}$$

so that $\operatorname{Log} L(s, \chi)$ is regular on the circle and, by Satz 6.1 of Prachar [7], we have

$$A(R) \leqslant C_{14} \log k \frac{\{\log\log(|t| + 3)\}^{9/4}}{\{\log(|t| + 3)\}^{3/4}} + C_{15}\{\log\log(|t| + 3)\}^3.$$

Also

$$|\operatorname{Log} L(s_0, \chi)| \leqslant \operatorname{Log} \frac{C_{16}}{\sigma_0 - 1},$$

and so by the Borel–Caratheodory theorem, in the region

$$\mathbf{R}s = \sigma \geqslant 1 - \frac{C_7}{M(k, |t| + 1)}, \qquad r \leqslant \frac{1}{2}(R + \sigma_0 - 1),$$

we have that

$$|\operatorname{Log} L(s, \chi)| \leqslant \frac{R + \sigma_0 - 1}{\frac{1}{2}(R - \sigma_0 + 1)} A(R) + \frac{3R + \sigma_0 - 1}{R - \sigma_0 + 1} \operatorname{Log} \frac{C_{16}}{\sigma_0 - 1}.$$

If $|t| \leqslant 2$ we set

$$\sigma_0 - 1 = \frac{2C_7}{M(k, |t| + 1)} = \frac{1}{2} R$$

and deduce

$$\begin{aligned} |\operatorname{Log} L(s, \chi)| &\leqslant 6A(R) + 7 \operatorname{Log} \frac{C_{16}}{\sigma_0 - 1} \\ &\leqslant C_9 \log k \frac{\{\log\log(|t| + 3)\}^{9/4}}{\{\log(|t| + 3)\}^{3/4}} + C_{10}\{\log\log(|t| + 3)\}^3 \\ &\quad + C_{11} \log\log 2k. \end{aligned}$$

Suppose now that $\chi \neq \chi_1$ and $|t| \leqslant 2$. In the region

$$\sigma \geqslant 1 - \frac{C_8(\epsilon)}{k^\epsilon}$$

we have

$$r \leqslant \frac{C_8(\epsilon)}{k^\epsilon} + \sigma_0 - 1, \qquad A(R) \leqslant C_{17} \log k,$$

and we set

$$\sigma_0 - 1 = \frac{1}{\log^2 k}.$$

Thus, again by the Borel–Caratheodory theorem,

$$|\operatorname{Log} L(s, \chi)| \leqslant C_{12} \log\log 2k.$$

Finally, let $\chi = \chi_1$. Then $\operatorname{Log} L(s, \chi)$ is regular in the region

$$\sigma \geqslant 1 - \frac{2C_8(\epsilon)}{k^\epsilon}, \qquad |t| \leqslant 2,$$

and we set

$$R = (\sigma_0 - 1) + \frac{2C_8(\epsilon)}{k^\epsilon}.$$

In the region

$$\sigma \geqslant 1 - \frac{\xi C_8(\epsilon)}{k^\epsilon}$$

we have

$$r \leqslant \sigma_0 - 1 + \frac{\xi C_8(\epsilon)}{k^\epsilon} \leqslant \frac{2\xi C_8(\epsilon)}{k^\epsilon}$$

if

$$\sigma_0 - 1 = \frac{\xi C_8(\epsilon)}{k^\epsilon}.$$

But $| L(s, \chi)| < C_{18} k^{1/2}$ in this region, and so

$$\begin{aligned} | \operatorname{Log} L(s, \chi)| &\leqslant \frac{4\xi}{2-\xi}\left(\frac{1}{2}\log k + C_{19}\right) + \frac{2+3\xi}{2-\xi}(\epsilon \log k + C_{20}(\xi, \epsilon)) \\ &\leqslant \frac{1}{2}\log k + C_{13}(\epsilon), \end{aligned}$$

if $\xi = \frac{1}{4}$ and $\epsilon \leqslant 3/11$. This completes the proof.

Proof of Theorem 4. As in the proof of Theorem 2, we have for $2\mathbf{R}y < 1$ and $| y | \leqslant 1$, $c = 1 + (1/\log x)$,

$$\begin{aligned} &\sum_{n<x} y^{\omega(n)} \sideset{}{'}\sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2 \\ &= \frac{1}{2i\pi}\int_{c-iT}^{c+iT} \frac{x^s}{s} F(s)\, ds + O\left(\frac{x^{1+\epsilon_1}}{T}\right) \end{aligned}$$

where

$$F(s) = \frac{1}{\varphi(k)} \prod_{p|k}\left(\frac{p^s}{p^s+y}\right) \frac{\zeta^2(s, y)}{\zeta(2s, y^2)} \sum_{\chi \neq \chi_0} L(s, y, \chi)\, L(s, y, \bar{\chi}).$$

We move the contour to

$$\sigma = 1 - \frac{C_{21}(\epsilon)}{M^*(k, | t | + 1)}$$

where

$$M^*(k, t) = \max\{k^\epsilon, \{\log(| t | + 3) \log\log(| t | + 3)\}^{3/4}\},$$

and

$$C_{21}(\epsilon) = \min\left(\frac{C_8(\epsilon)}{4}, C_7\right),$$

except for a lacet around $s = 1$. The contour is completed by horizontal lines at $| t | = T$. We have

$$\frac{1}{2i\pi}\int_{c-iT}^{c+iT} \frac{x^s}{s} F(s)\, ds = \frac{1}{2i\pi}\int_{\Gamma_1} \frac{x^s}{s} F(s)\, ds + \frac{1}{2i\pi}\int_{\Gamma_2} \frac{x^s}{s} F(s)\, ds,$$

where Γ_1 denotes the lacet, Γ_2 the rest of the contour. Next,

$$L(s, y, \chi) = \{L(s, \chi)\}^y H(s, y, \chi)$$

where $H(s, y, \chi)$ is regular and bounded independently of k for $\sigma > \frac{1}{2} + \delta$, and $| y | \leqslant 1$. Thus

$$| L(s, y, \chi)| \ll \exp\{| y | \, | \operatorname{Log} L(s, \chi)|\}$$

and, by Lemma 3, on Γ_2 we have

$$F(s) \leqslant C_{22}(\epsilon) \exp\{C_{23} \log k + C_{24}(\log \log T)^3\},$$

since for small t,

$$\zeta(s) = O(k^\epsilon).$$

We do not use the full strength of the lemma since the error which arises in the estimation of the integral Γ_1 is more serious than the contribution of that on Γ_2. We find that

$$\frac{1}{2i\pi} \int_{\Gamma_2} \frac{x^s}{s} F(s)\, ds \leqslant \left\{\frac{x^c}{T} + x \log T \exp\left(-\frac{C_{21}(\epsilon) \log x}{M^*(k, T+1)}\right)\right\}$$
$$\cdot\, C_{22}(\epsilon) \exp\{C_{23} \log k + C_{24}(\log \log T)^3\}.$$

Thus setting $T = x$, we have that the left-hand side does not exceed

$$C_{25}(\epsilon)\, x \exp\left\{-C_{26}(\epsilon) \frac{(\log x)^{1/4}}{(\log \log x)^{3/4}}\right\},$$

provided

$$k^\epsilon < (\log x \cdot \log \log x)^{3/4}.$$

It remains to consider the integral around the lacet Γ_1. We set

$$F(s) = \frac{1}{(s-1)^{2y}} G(s)$$

and define

$$\frac{1}{(s-1)^{2y}} = \exp\{-2y(\log | s - 1 | + i \arg(s - 1))\},$$

where the argument is zero on the positive real axis. Consider first the integral around part of a circle of radius δ around $s = 1$. Setting

$$s - 1 = \delta e^{i\theta}$$

we have

$$\frac{1}{2i\pi}\int \frac{x^s}{s}F(s)\,ds \ll \int_{-\pi+\alpha}^{\pi-\alpha} \frac{\delta x^{1+\delta}}{1-\delta}\exp\{-2\mathbf{R}y\log\delta + 2\mathbf{I}y\theta\}\,d\theta$$
$$\ll \delta^{1-2\mathbf{R}y}$$

where here the $\ll$ depends on x and G. If $2\mathbf{R}y < 1$ we conclude that this integral tends to zero with δ. We next examine the integral along the upper and lower section of the lacet from $1 - (C_{21}(\epsilon)/k^\epsilon)$ to 1. Letting the lacet collapse onto the real line and $\delta \to 0$ together, we have

$$\frac{1}{2i\pi}\int_{\Gamma_1}\frac{x^s}{s}F(s)\,ds = \frac{1}{2i\pi}\int_{1-\omega_0}^{1}\frac{x^s}{s}G(s)\exp\{-2y(\log(1-s) - i\pi\}\,ds$$
$$-\frac{1}{2i\pi}\int_{1-\omega_0}^{1}\frac{x^s}{s}G(s)\exp\{-2y(\log(1-s) + i\pi\}\,ds,$$

and therefore, for $\epsilon_1 = \frac{1}{2}$ say,

$$\sum_{n<x} y^{\omega(n)} \sum_{\substack{h=1\\(h,k)=1}}^{k}\left(\tau(n;k,h) - \frac{\tau(n;k)}{\varphi(k)}\right)^2$$
$$= \frac{\sin 2\pi y}{\pi}\int_0^{\omega_0}\frac{x^{1-\omega}}{1-\omega}G(1-\omega)\,\omega^{-2y}\,d\omega$$
$$+ O\left(C_{25}(\epsilon)\,x\exp\left\{-C_{26}(\epsilon)\frac{(\log x)^{1/4}}{(\log\log x)^{3/4}}\right\}\right),$$

where

$$\omega_0 = C_{21}(\epsilon)\,k^{-\epsilon}$$

Next, by Cauchy's theorem,

$$\frac{G(1-\omega)}{1-\omega} = G(1) + \frac{1}{2i\pi}\int_D \frac{-\omega G(z)\,dz}{z(z-1)(z-1+\omega)}.$$

Take D to be a circle, centre $1 - \frac{1}{2}\omega_0$ and radius ω_0. $G(z)$ is regular in this region and since

$$(s-1)^{2y}\frac{\zeta^2(s,y)}{\zeta(2s,y^2)}\prod_{p|k}\left(\frac{p^s}{p^s+y}\right) \ll k^{\epsilon_1},$$

and

$$\frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} L(s, y, \chi) L(s, y, \bar{\chi})$$

$$\ll \frac{1}{\varphi(k)} \exp\{2 \mid y \mid \mid \operatorname{Log} L(s, \chi_1) \mid\}$$

$$+ \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0 \text{ or } \chi_1} \exp\{\mid y \mid (\mid \operatorname{Log} L(s, \chi) \mid + \mid \operatorname{Log} L(s, \bar{\chi}) \mid)\}$$

$$\ll \exp\{2C_{12} \log \log 2k\} + \frac{1}{\varphi(k)} \exp\{\mid y \mid \log k + 2C_{13}(\epsilon)\}$$

$$\ll C_{27}(\epsilon)(\log k)^{2C_{12}},$$

then

$$\mid G(z) \mid \leqslant C_{28}(\epsilon)\, k^{2\epsilon_1}$$

on the contour of integration.

It follows that

$$\frac{G(1-\omega)}{1-\omega} = G(1) - \omega K(\omega)$$

for $0 \leqslant \omega \leqslant \omega_0$, with

$$\mid K(\omega) \mid \leqslant C_{30}(\epsilon)\, k^{2\epsilon}.$$

Thus

$$\int_0^{\omega_0} \frac{x^{1-\omega}}{1-\omega} G(1-\omega)\, \omega^{-2y}\, d\omega$$

$$= G(1) \int_0^{\omega_0} x^{1-\omega} \omega^{-2y}\, d\omega + O\left(C_{30}(\epsilon)\, k^{2\epsilon} \int_0^{\omega_0} x^{1-\omega} \omega^{1-2\mathbf{R}y}\, d\omega\right)$$

$$= \frac{G(1)x}{(\log x)^{1-2y}} \int_0^{\omega_0 \log x} e^{-u} u^{-2y}\, du + O\left(\frac{C_{31}(\epsilon)\, k^{2\epsilon} x}{(\log x)^{2-2\mathbf{R}y}}\right)$$

$$= \frac{G(1)\, x\Gamma(1-2y)}{(\log x)^{1-2y}} (1 + O(x^{-\omega_0/2})) + O\left(\frac{C_{31}(\epsilon)\, k^{2\epsilon} x}{(\log x)^{2-2\mathbf{R}y}}\right),$$

and

$$\sum_{n<x} y^{\omega(n)} \sum_{\substack{h=1 \\ (h,k)=1}}^{k} \left(\tau(n; k, h) - \frac{\tau(n; k)}{\varphi(k)}\right)^2$$

$$= \frac{G(1)x}{\Gamma(2y)(\log x)^{1-2y}} (1 + O(x^{-\omega_0/2})) + O\left(\frac{C_{31}(\epsilon)\, x k^{2\epsilon}}{(\log x)^{2-2\mathbf{R}y}}\right)$$

$$+ O\left(C_{25}(\epsilon) x \exp\left\{\frac{-C_{26}(\epsilon)(\log x)^{1/4}}{(\log \log x)^{3/4}}\right\}\right).$$

Also

$$G(1) = H(y) \prod_{p|k} \left(\frac{p}{p+y}\right) \frac{1}{\varphi(k)} \sum_{\chi \neq \chi_0} L(1, y, \chi)\, L(1, y, \bar{\chi})$$
$$= O(\log^3 k).$$

Since $1/\Gamma(2y)$ is bounded for the y under consideration, the most serious error term is the second one. This completes the proof.

LEMMA 4. For $\alpha > 0$, if $\nu(n)$ denotes the number of distinct prime factors of n, we have that

$$\sum_{n \leqslant x} \frac{2^{(1-\alpha)\omega(n)}}{2^{\nu(n)}} \ll x/\alpha.$$

Proof. Let

$$T_\alpha(x) = \sum_{\substack{n \leqslant x \\ 2 \nmid n}} \frac{2^{(1-\alpha)\omega(n)}}{2^{\nu(n)}} \leqslant T_0(x).$$

It is not difficult to prove that

$$T_0(x) \sim \frac{x}{2} \prod_{p>2} \left(1 + \frac{1}{p(p-2)}\right),$$

and hence

$$\sum_{n \leqslant x} \frac{2^{(1-\alpha)\omega(n)}}{2^{\nu(n)}} = T_\alpha(x) + \frac{2^{1-\alpha}}{2} T_\alpha\left(\frac{x}{2}\right) + \frac{4^{1-\alpha}}{2} T_\alpha\left(\frac{x}{4}\right) + \cdots$$
$$\ll x\{1 + \tfrac{1}{2}(2^{-\alpha} + 4^{-\alpha} + 8^{-\alpha} + \cdots)\}$$
$$\ll x/\alpha.$$

This completes the proof.

Proof of Theorem 5. Let $S_k(\eta)$ denote the set of integers n for which, for at least one value of h prime to k, one of the inequalities

$$(1 - \eta) \frac{\tau(n; k)}{\varphi(k)} < \tau(n; k, h) < (1 + \eta) \frac{\tau(n; k)}{\varphi(k)}$$

does not hold. Let $\nu_k(n)$ denote the number of distinct prime factors of n prime to k. Then,

$$\sum_{\substack{n \leqslant x \\ n \in S_k(\mu)}} 1 \leqslant \sum_{\substack{n \leqslant x \\ \nu_k(n) < t}} 1 + \sum_{\substack{n \leqslant x \\ n \in S_k(\eta) \\ \nu_k(n) \geqslant t}} 1,$$

and we denote the first sum on the right by Σ_1. By the Cauchy–Schwarz inequality, for real $\alpha > 0$,

$$\frac{1}{x}\Bigg(\sum_{\substack{n\leqslant x\\ n\in S_k(\eta)\\ \nu_k(n)\geqslant t}} 1\Bigg)^2 \leqslant \frac{1}{x}\Bigg(\sum_{\substack{n\leqslant x\\ n\in S_k(\eta)\\ \nu_k(n)\geqslant t}} \frac{2^{\nu(n)}}{2^{(1-\alpha)\omega(n)}}\Bigg)\Bigg(\sum_{n\leqslant x} \frac{2^{(1-\alpha)\omega(n)}}{2^{\nu(n)}}\Bigg)$$

$$\ll \frac{1}{\alpha} \sum_{\substack{n\leqslant x\\ n\in S_k(\eta)\\ \nu_k(n)\geqslant t}} \frac{2^{\nu(n)}}{2^{(1-\alpha)\omega(n)}} = \frac{1}{\alpha}\Sigma_2,$$

say. Here we have used Lemma 4. Set

$$t = (1-\omega)\log\log x,$$

so that

$$t+\nu(k) = (1-\omega)\log\log x + O\left(\frac{\log\log x}{\log\log\log x}\right)$$
$$= (1-\omega_1)\log\log x,$$

where $4\omega_1{}^2 \geqslant 3\omega^2$ provided ω exceeds some absolute multiple of $1/\log\log\log x$. Thus

$$\Sigma_1 \leqslant \sum_{\substack{n\leqslant x\\ \nu(n)\leqslant t+\nu(k)}} 1 \ll x/(\log x)^{\omega_1{}^2/3}.$$

The inequality may also be deduced from the classical results of Hardy & Ramanujan [4], or from the general theory of additive functions (see, for example, Kubilius [5]). Next, for

$$k \leqslant 2^{\frac{1}{2}\log\log x},$$

we may apply Theorem 4 with $\epsilon = 3/11$, $y = 1/2^{1-\alpha}$. We have

$$\sum_{\substack{n\leqslant x\\ \nu_k(n)\geqslant t\\ n\in S_k(\eta)}} \frac{1}{2^{(1-\alpha)\omega(n)}}\left(\tau(n;k,h) - \frac{\tau(n;k)}{\varphi(k)}\right)^2 \ll x(\log x)^{2\alpha-1}\log^2 k.$$

The squared expression on the left is at least

$$\eta^2\frac{\tau^2(n;k)}{\varphi^2(k)} \geqslant \frac{\eta^2}{\varphi^2(k)}4^{\nu_k(n)} \geqslant \frac{\eta^2 2^t\cdot 2^{\nu_k(n)}}{\varphi^2(k)},$$

and therefore

$$\sum_{\substack{n\leqslant x\\ \nu_k(n)\geqslant t\\ n\in S_k(\eta)}} \frac{2^{\nu(n)}}{2^{(1-\alpha)\omega(n)}} \leqslant \frac{xk^2 \log^2 k \cdot 2^{\nu(k)}(\log x)^{2^\alpha-1}}{\eta^2 2^t}.$$

If

$$k^2 = 2^{(1-\epsilon)\log\log x},$$

the sum on the left does not exceed

$$C_{32}x(\log\log x)^2\,\eta^{-2}\exp\left\{\log 2\left(\nu(k) + \left\{\omega - \epsilon + \frac{2^\alpha - 1}{\log 2}\right\}\log\log x\right)\right\},$$

and we now set $\omega = \epsilon/2$, $\alpha = \epsilon/10$; note that we require

$$\omega \geqslant \omega_0/\log\log\log x,$$

and since

$$\nu(k) = O\left(\frac{\log\log x}{\log\log\log x}\right),$$

we take $\epsilon \geqslant \epsilon_0/\log\log\log x$. Thus

$$\Sigma_2 \leqslant \frac{C_{33}x}{\eta^2(\log x)^{\epsilon/4}}.$$

Together with the upper bound for Σ_1 this completes the proof.

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